HIDDEN SYMMETRY OF THE DIFFERENTIAL CALCULUS ON THE QUANTUM MATRIX SPACE

S. Sinel'shchikov* L. Vaksman†

Mathematics Department, Institute for Low Temperature Physics and Engineering, 47 Lenin Avenue, 310164 Kharkov, Ukraine

1. This work solves a problem whose simple special case occurs in a construction of a quantum unit ball of $\mathbb{C}^n$ (in the spirit of [10]). Within the framework of that theory, the automorphism group of the ball $SU(n,1) \subset SL(n+1)$ is essential. The problem is that the Wess-Zumino differential calculus in quantum $\mathbb{C}^n$ [11] at a first glance seems to be only $U_q\mathfrak{sl}_n$-invariant. In that particular case the lost $U_q\mathfrak{sl}_{m+n}$-symmetry can be easily detected. The main result of this work is in disclosing the hidden $U_q\mathfrak{sl}_n$-symmetry for bicovariant differential calculus in the quantum matrix space $\mathrm{Mat}(m,n)$. (Note that for $n=1$ we have the case of a ball).

The aurhors are grateful to V. Akulov and G. Maltsiniotis for a helpful discussion of the results.

2. We start with recalling the definition of the Hopf algebra $U_q\mathfrak{sl}_N$, $N>1$, over the field $\mathbb{C}(q)$ of rational functions of an indeterminate q [4], [5]. (We follow the notations of [3]).

For $i,j \in \{1,\ldots,N-1\}$ let

$$a_{ij} = \begin{cases} 2, & i-j=0 \\ -1, & |i-j|=1 \\ 0, & |i-j|>1. \end{cases}$$

The algebra $U_q\mathfrak{sl}_N$ is defined by the generators $\{E_i,\ F_i,\ K_i,\ K_i^{-1}\}$ and the relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1$$

$$K_iE_j = q^{a_{ij}}E_jK_i, \quad K_iF_j = q^{-a_{ij}}F_jK_i$$

$$E_iF_j - F_jE_i = \delta_{ij}(K_i - K_i^{-1})/(q-q^{-1})$$

$$E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0, \quad |i-j| = 1$$

*Partially supported by the ISF grant U2B200

†Partially supported by the AMS fSU grant and by ISF grant U21200

$$F_i^2 F_j \,-\, (q+q^{-1}) F_i F_j F_i \,+\, F_j F_i^2 \,=\, 0, \quad |i-j| \,=\, 1$$
$$[E_i, E_j] \,=\, [F_i, F_j] \,=\, 0, \quad |i-j| \neq 1.$$

A comultiplication Δ, an antipode S and a counit ε are defined by

$$\Delta E_i \,=\, E_i \otimes 1 \,+\, K_i \otimes E_i, \quad \Delta F_i \,=\, F_i \otimes K_i^{-1} \,+\, 1 \otimes F_i,$$
$$\Delta K_i \,=\, K_i \otimes K_i, \quad S(E_i) \,=\, -K_i^{-1} E_i,$$
$$S(F_i) \,=\, -F_i K_i, \quad S(K_i) \,=\, K_i^{-1},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

3. Remind a description of a differential algebra $\Omega^*(\mathrm{Mat}(m,n))_q$ on a quantum matrix space [2] [8].

Let $i, j, i', j' \in \{1, 2, \ldots, m+n\}$, and

$$\check{R}_{ij}^{i'j'} \;=\; \begin{cases} q^{-1}, & i = j = i' = j' \\ 1, & i' = j \text{ and } j' = i \text{ and } i \neq j \\ q^{-1} - q, & i = i' \text{ and } j = j' \text{ and } i < j \\ 0, & \text{otherwise} \end{cases}$$

$\Omega^*(\mathrm{Mat}(m,n))_q$ is given by the generators $\{t_a^\alpha\}$ and relations

$$\sum_{\gamma,\delta} \check{R}_{\gamma\delta}^{\alpha\beta} t_a^\gamma t_b^\delta \;=\; \sum_{c,d} \check{R}_{ab}^{cd} t_d^\beta t_c^\alpha$$
$$\sum_{a',b',\gamma',\delta'} \check{R}_{\gamma'\delta'}^{\alpha\beta} \check{R}_{ab}^{a'b'} t_{a'}^{\gamma'} dt_{b'}^{\delta'} \;=\; dt_a^\alpha t_b^\beta$$
$$\sum_{a',b',\gamma',\delta'} \check{R}_{\gamma'\delta'}^{\alpha\beta} \check{R}_{ab}^{a'b'} dt_{a'}^{\gamma'} dt_{b'}^{\delta'} \;=\; -dt_a^\alpha dt_b^\beta$$

$(a, b, c, d, a', b' \in \{1, \ldots, n\}; \quad \alpha, \beta, \gamma, \delta, \gamma', \delta' \in \{1, \ldots, m\})$.

Let us define a grading by $\deg(t_a^\alpha) = 0$, $\deg(dt_a^\alpha) = 1$. With that, $\mathbb{C}[\mathrm{Mat}(m,n)]_q \;=\; \Omega^0(\mathrm{Mat}(m,n)))_q$ will stand for a subalgebra of zero degree elements .

4. Let A be a Hopf algebra and F an algebra with unit and an A-module the same time. F is said to be a A-module algebra [1] if the multiplication $m : F \otimes F \to F$ is a morphism of A-modules, and $1 \in F$ is an invariant (that is
$a(f_1 f_2) \;=\; \sum_j a_j' f_1 \otimes a_j'' f_2, \quad a1 \;=\; \varepsilon(a)1$ for all $a \in A$; $f_1, f_2 \in F$, with $\Delta(a) \;=\; \sum_j a_j' \otimes a_j''$).

An important example of an A-module algebra appears if one supplies A^* with the structure of an A-module: $\langle af, b\rangle \;=\; \langle f, ba\rangle$, $a, b \in A$, $f \in A^*$.

5. Our immediate goal is to furnish $\mathbb{C}[\mathrm{Mat}(m,n)]_q$ with a structure of a $U_q\mathfrak{sl}_{m+n}$-module algebra via an embedding $\mathbb{C}[\mathrm{Mat}(m,n)]_q \hookrightarrow (U_q\mathfrak{sl}_{m+n})^*$.

Let $\{e_{ij}\}$ be a standard basis in $\mathrm{Mat}(m+n)$ and $\{f_{ij}\}$ the dual basis in $\mathrm{Mat}(m+n)^*$. Consider a natural representation π of $U_q\mathfrak{sl}_{m+n}$:

$$\pi(E_i)\;=\;e_{i\,i+1},\quad \pi(F_i)\;=\;e_{i+1\,i},\quad \pi(K_i)\;=\;qe_{ii}+q^{-1}e_{i+1\,i+1}+\sum_{j\neq i,i+1}e_{jj}.$$

The matrix elements $u_{ij}\;=\;f_{ij}\pi\in(U_q\mathfrak{sl}_{m+n})^*$ of the natural representation may be treated as "coordinates" on the quantum group SL_{m+n} [4]. To construct "coordinate" functions on a big cell of the Grassmann manifold, we need the following elements of $\mathbb{C}[\mathrm{Mat}(m,n)]_q$

$$x(j_1,j_2,\ldots,j_m)\;=\;\sum_{w\in S_m}(-q)^{l(w)}u_{1j_{w(1)}}u_{2j_{w(2)}}\ldots u_{mj_{w(m)}},$$

with $1\leq j_1<j_2<\ldots<j_m\leq m+n$, and $l(w)\;=\;\mathrm{card}\{(a,b)|\;a<b \text{ and } w(a)>w(b)\}$ being the "length" of a permutation $w\in S_m$.

Proposition 1. *$x(1,2,\ldots,m)$ is invertible in $(U_q\mathfrak{sl}_{m+n})^*$, and the map*

$$t_a^\alpha\mapsto x(1,2,\ldots,m)^{-1}x(1,\ldots,\widehat{m+1-\alpha},\ldots,m,m+a)$$

can be extended up to an embedding

$$i:\;\mathbb{C}[Mat(m,n)]_q\hookrightarrow(U_q\mathfrak{sl}_{m+n})^*.$$

(The sign $\hat{}$ here indicates the item in a list that should be omitted).

Proposition 1 allows one to equip $\mathbb{C}[\mathrm{Mat}(m,n)]_q$ with the structure of a $U_q\mathfrak{sl}_{m+n}$-module algebra :

$$i\xi t_a^\alpha\;=\;\xi it_a^\alpha,\qquad \xi\in U_q\mathfrak{sl}_{m+n},\;a\in\{1,\ldots,n\},\;\alpha\in\{1,\ldots,m\}.$$

6. The main result of our work is the following

Theorem 1. *$\Omega^*(Mat(m,n))_q$ admits a unique structure of a $U_q\mathfrak{sl}_{m+n}$-module algebra such that the embedding*

$$i:\;\mathbb{C}[Mat(m,n)]_q\hookrightarrow\Omega^*(Mat(m,n))_q$$

and the differential

$$d:\;\Omega^*(Mat(m,n))_q\;\rightarrow\;\Omega^*(Mat(m,n))_q$$

are the morphisms of $U_q\mathfrak{sl}_{m+n}$-modules.

REMARK 1. The bicovariance of the differential calculus on the quantum matrix space allows one to equip the algebra $\Omega^*(\mathrm{Mat}(m,n))_q$ with a structure of $U_q\mathfrak{s}(\mathfrak{gl}_m \times \mathfrak{gl}_n)$-module, which is compatible with multiplication in $\Omega^*(\mathrm{Mat}(m,n))_q$ and differential d. Theorem 1 implies that $\Omega^*(\mathrm{Mat}(m,n))_q$ possess an additional hidden symmetry since $U_q\mathfrak{sl}_{m+n} \supsetneqq U_q\mathfrak{s}(\mathfrak{gl}_m \times \mathfrak{gl}_n)$.

REMARK 2. Let $q_0 \in \mathbb{C}$ and q_0 is not a root of unity. It follows from the explicit formulae for $E_m t_a^\alpha$, $F_m t_a^\alpha$, $K_m^{\pm 1} t_a^\alpha$, $\quad a \in \{1,\ldots,n\}$, $\quad \alpha \in \{1,\ldots,m\}$, that the "specialization" $\Omega^*(\mathrm{Mat}(m,n))_{q_0}$ is a $U_{q_0}\mathfrak{sl}_{m+n}$-module algebra.

7. Supply the algebra $U_q\mathfrak{sl}_{m+n}$ with a grading as follows:

$$\deg(K_i) = \deg(E_i) = \deg(F_i) = 0, \quad \text{for} \quad i \neq m,$$

$$\deg(K_m) = 0, \quad \deg(E_m) = 1, \quad \deg(F_m) = 0.$$

The proofs of Proposition 1 and Theorem 1 reduce to the construction of graded $U_q\mathfrak{sl}_{m+n}$-modules which are dual respectively to the modules of functions $\Omega^0(\mathrm{Mat}(m,n))_q$ and that of 1-forms $\Omega^1(\mathrm{Mat}(m,n))_q$. The dual modules are defined by their generators and correlations. While proving the completeness of the correlation list we implement the "limit specialization" $q_0 = 1$ (see [3], p. 476).

The passage from the order one differential calculus $\Omega^0(\mathrm{Mat}(m,n))_q \xrightarrow{d} \Omega^1(\mathrm{Mat}(m,n))_q$ to $\Omega^*(\mathrm{Mat}(m,n))_q$ is done via a universal argument described in a paper by G. Maltsiniotis [9]. This argument doesn't break $U_q\mathfrak{sl}_{m+n}$-symmetry.

8. Our approach to the construction of order one differential calculus is completely analogous to that of V. Drinfeld [4] used initially to produce the algebra of functions on a quantum group by means of a universal enveloping algebra.

9. The space of matrices is the simplest example of an irreducible prehomogeneous vector space of parabolic type [7]. Such space can be also associated to a pair constituted by a Dynkin diagram of a simple Lie algebra $\mathcal{G}$ and a distinguished vertex of this diagram. Our method can work as an efficient tool for producing $U_q\mathcal{G}$-invariant differential calculi on the above prehomogeneous vector spaces.

Note that $U_q\mathcal{G}$-module algebras of polynomials on quantum prehomogeneous spaces of parabolic type were considered in a recent work of M. S. Kebe [6].

Acknowledgement. The authors would like to express their gratitude to Prof. A. Boutet de Monvel at University Paris VII for the warm hospitality during the work on this paper.

References

[1] E. Abe. Hopf Algebras, Cambridge Tracts in Mathematics 74, Cambridge University Press, Cambridge, 1980.

[2] V. Chari and A. Pressley. A guide to quantum groups. Cambridge Univ. Press, 1994.

[3] C. De Concini and V. G. Kac, *Representations of quantum groups at roots of* 1, in Operator Algebras, Unitary representations, Enveloping Algebras and Invariant Theory, A. Connes, M. Duflo, A. Joseph, R. Rentschler (eds.), pp. 471- 506, Birkhäuser, Boston, 1990.

[4] V. G. Drinfel'd , *Quantum groups*, in Proceedings of International Congress of Mathematicians, Berkeley, 1986, A. M. Gleason (ed.), pp. 798 - 820, American Mathematical Society, Providence, RI, 1987.

[5] M. Jimbo, *Quantum R-matrix related to the generalized Toda system: an algebraic approach*, in Field Theory, Quantum Gravity and Strings, H. J. de Vega & N. Sanchez (eds.), Lecture Notes in Phys., **246**, 335 - 361, Springer, Berlin,1986.

[6] Mâlek Stefan Kebe, *$\mathcal{O}$-algèbres quantiques*, C. R. Acad. Sci. Paris, **322** (1996), Série 1, 1 - 4.

[7] T. Kimura, *A classification theory of Prehomogeneous vector spaces*, Advanced Studies in Pure Mathematics **14**, 1988, Representations of Lie groups, Kyoto, Hiroshima, 1986, p. 223 - 256.

[8] G. Maltsiniotis, *Groupes quantiques et structures différentielles*, C. R. Acad. Sci., Paris, Série I, **311** (1990), 831 - 834.

[9] G. Maltsiniotis, *Le langage des espaces et des groupes quantiques*, Comm. Math. Phys., **151** (1993), 275 - 302.

[10] W. Rudin. Function Theory in the Unit Ball of $\mathbb{C}^n$. Springer – Verlag, N.-Y., 1980.

[11] J. Wess and B. Zumino, *Covariant differential calculus on the quantum hyperplane*, Nucl. Phys. B, Proc. Suppl. **18 B**, 1991, 302 - 312.